Int. J. Solids Structures, 1972, Vol. 8, pp. 1413 to 1431. Pergamon Press. Printed in Great Britain

FINITE ELEMENT LIMIT ANALYSIS USING LINEAR PROGRAMMING

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Abstract—This paper concerns the development of numerical methods for the determination of the ultimate load of two- and three-dimensional structures assuming an ideal rigid-plastic behavior of the material. According to the classical plasticity theory the ultimate load problem can be mathematically formulated as the problem of finding the maximum or minimum of a linear function, whose independent variables are subjected to inequality constraints. In order to use linear programming techniques these generally nonlinear constraints are approximated by sets of linear inequality restrictions. Linear equilibrium or kinematic compatibility equations have then to be formulated, the corresponding coefficients being determined by virtual work methods. This requires the assumption of parametric stress and displacement fields, which are constructed by mean of finite element procedures. Numerical results for two plate-bending models are presented and some possible plate-stretching models are described and discussed.

1. INTRODUCTION

THE behavior of complex structural systems above the elastic range can be determined by a nonlinear type of analysis. However, the description of the nonlinear material properties in mathematical terms often cannot be accurate enough as to justify the great computational effort generally involved in such procedures. This is true for most soil and rock mechanics as well as for many reinforced concrete ultimate load problems.

On the other hand, if a rigid-ideal-plastic material behavior can be assumed, the upper-bound and the lower-bound theorems of the plasticity theory provide a powerful tool for the direct determination of limit loads. Of course, the scope of such an analysis is somehow limited as the only informations obtained are a load factor and possibly the shape of the collapse mechanism. Moreover, yield-conditions, just like nonlinear stress–strain relations, are sometimes difficult to determine in an accurate way.

However, if it is possible to develop easy-to-use computer programs capable of treating a wide variety of problems with a limited computational effort, plastic analysis can find many applications in everyday's civil engineering practice.

The aim of this paper is to show how to assume mathematical models for two- and three-dimensional structures leading to such efficient computer programs. The latest developments of the finite element method, especially those connected with "mixed" formulation of elastic analysis [4, 5, 8, 1, 2], are used for this purpose. The limit load is then found by linear programming.

Formally the problem can be stated as that of finding the load factor λ for which a given structure collapses (see Fig. 1). The loads are body forces λg_i and surface tractions λt_i acting on the S_t-portion of the external boundary surface in the direction of the cartesian coordinates $x_1, x_2, x_3 (i = 1, 2, 3)$. On the S_u-portion of the surface S ($S = S_t + S_u$) the

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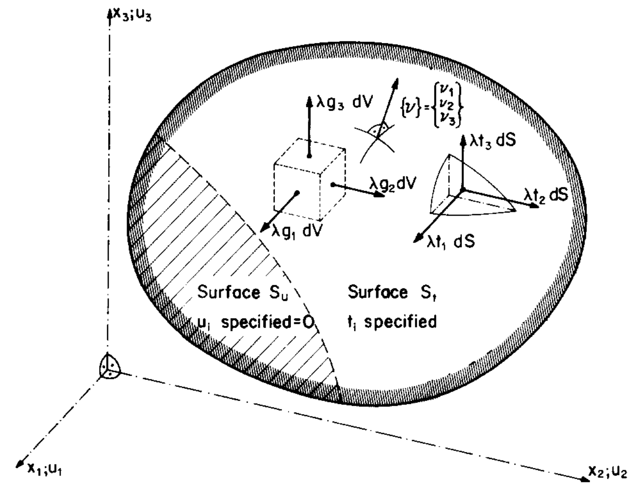

FIG. 1. Three-dimensional continuum loaded by body-forces λg_i and surface-tractions λt_i.

displacements u_i must be zero. Corresponding stresses and strains are denoted by σ_{ij} and ε_{ij} $(i = 1, 2, 3; j = 1, 2, 3)$.

Throughout the paper index- and matrix-notation are used. Index notation always refers to the stated three-dimensional problem in cartesian coordinates. A repeated subscript means summation over all possible values of the subscript. Contrary to standard tensor notation, however, this sum convention applies only for subscripts, not for superscripts. A comma followed by one or two subscripts means partial derivative with respect to the corresponding coordinates.

Matrix and vector symbols are always written in brackets. As matrix notation can be applied to any kind of structures, matrices and vectors are generally not formally defined. It will not be difficult for the experienced reader to find the definitions applying to his particular problem.

The procedures suggested generally do not provide a mathematical bound of the true value of the load factor λ but just a good approximation of it, as some of the requirements of the lower-bound and of the upper-bound theorem of plasticity are not satisfied exactly. Only in special cases is it possible to show that the obtained value of λ must be a mathematical bound of the true value.

2. LINEARISATION OF THE YIELD CONDITIONS

The stress components σ_{ij} within an ideal rigid-plastic body must everywhere satisfy the yield condition

$$0 \leq c - f(\sigma_{ij}) \tag{1}$$

where both the positive constant c and the function $f(\sigma_{ij})$ are material dependent. The yield condition (1) will be checked only at a finite number of points with coordinates $x_1^q, x_2^q, x_3^q (q = 1\text{–}Q)$, where the stress components assume the values σ_{ij}^q:

$$0 \leq c^q - f^q(\sigma_{ij}^q) \qquad (q = 1\text{–}Q) \tag{2}$$

The equation:

$$0 = c^q - f^q(\sigma^q_{ij}) \tag{3}$$

describes, in the σ^q_{ij}-space, the yield surface at a checkpoint q (see Fig. 2). In order to use

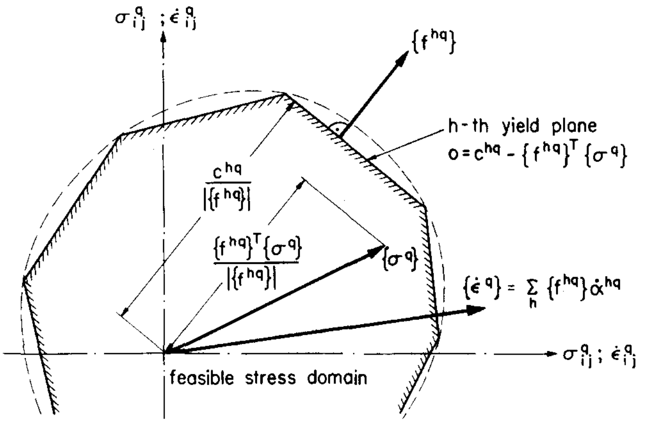

FIG. 2. Section through the linearized yield surface in the σ^q_{ij}; $\dot\varepsilon^q_{ij}$-space at a checkpoint q.

linear programming algorithms each of the nonlinear inequalities (2) has to be approximated by a set of linear inequalities:

$$0 \leq c^{hq} - f^{hq}_{ij}\sigma^q_{ij} \qquad (h = 1\text{–}H^q). \tag{4}$$

In matrix notation:

$$0 \leq c^{hq} - \{f^{hq}\}^T\{\sigma^q\} \qquad (h = 1\text{–}H^q) \tag{5}$$

or:

$$0 \leq \{c^q\} - [f^q]^T\{\sigma^q\}. \tag{6}$$

The yield surface is approximated by a polyhedron, whose faces are given by the equations:

$$0 = c^{hq} - f^{hq}_{ij}\sigma^q_{ij} = c^{hq} - \{f^{hq}\}^T\{\sigma^q\} \qquad (h = 1\text{–}H^q). \tag{7}$$

A vector $\{f^{hq}\}$ is normal to the hth face of the qth yield polyhedron and pointed toward the outside of the admissible stress domain. The inequalities (5) state, that the yield conditions at the checkpoint q are satisfied, if the projection of the stress vector on each of the $\{f^{hq}\}$-vectors is not greater than $c^{hq}/|\{f^{hq}\}|$ ($c^{hq} > 0$).

At least for three-dimensional stress spaces (for instance in plate-stretching and plate-bending problems) the approximation of the nonlinear yield conditions by sets of linear inequalities is generally easy. For the reinforced concrete plate-bending models described in Section 9 the following nonlinear yield conditions (see Wolfensberger [12]) are to be linearized:

$$\begin{gathered} P_x \geq m_x \geq N_x \\ P_y \geq m_y \geq N_y \\ (P_x - m_x)(P_y - m_y) \geq (m_{xy} - P_{xy})^2 \\ (N_x + m_x)(N_y + m_y) \geq (m_{xy} - N_{xy})^2, \end{gathered} \tag{8}$$

where m_x, m_y, m_{xy} are the bending and twisting moments, defined in the usual way. P_x, P_y, N_x, N_y are the positive and negative yield moments in x- and y-direction and P_{xy} and N_{xy} are constants, which are only different from zero, if the directions of the reinforcing steel bars are different from the directions x and y. Wolfensberger [12] suggests the linearization of the yield conditions (8) by eight linear inequalities leading to the following definition of the vectors $\{c^q\}$, $\{\sigma^q\}$ and of the matrix $[f^q]^T$:

$$\{c^q\} \equiv \begin{Bmatrix} P_x+P_{xy} \\ P_x-P_{xy} \\ P_y+P_{xy} \\ P_y-P_{xy} \\ N_x+N_{xy} \\ N_x-N_{xy} \\ N_y+N_{xy} \\ N_y-N_{xy} \end{Bmatrix} \quad [f^q]^T \equiv \begin{bmatrix} +1 & 0 & +1 \\ +1 & 0 & -1 \\ 0 & +1 & +1 \\ 0 & +1 & -1 \\ -1 & 0 & +1 \\ -1 & 0 & -1 \\ 0 & -1 & +1 \\ 0 & -1 & -1 \end{bmatrix} \quad \{\sigma^q\} \equiv \begin{Bmatrix} m_x^q \\ m_y^q \\ m_{xy}^q \end{Bmatrix}. \tag{9}$$

Other examples of linearized yield conditions can be found in Ref. [3].

3. INTERNAL RATE OF DISSIPATION

According to the plasticity theory the stress components σ_{ij} and the corresponding strain velocity components $\dot{\varepsilon}_{ij}$ during collapse are related as follows:

$$\dot{\varepsilon}_{ij} = \frac{\partial f}{\partial \sigma_{ij}} \dot{\alpha} \begin{cases} \dot{\alpha} \geq 0 & \text{if} \quad 0 = c - f(\sigma_{ij}) \\ \dot{\alpha} = 0 & \text{if} \quad 0 < c - f(\sigma_{ij}). \end{cases} \tag{10}$$

In a region of the rigid-plastic body with volume ΔV^q, where the yield surface is approximated by the qth yield polyhedron, equation (10) becomes:

$$\dot{\varepsilon}_{ij} = \sum_h f_{ij}^{hq} \dot{\alpha}^{hq} \begin{cases} \dot{\alpha}^{hq} \geq 0 & \text{if} \quad 0 = c^{hq} - f_{ij}^{hq} \sigma_{ij} \\ \dot{\alpha}^{hq} = 0 & \text{if} \quad 0 < c^{hq} - f_{ij}^{hq} \sigma_{ij}. \end{cases} \tag{11}$$

Or, in matrix notation:

$$\{\dot{\varepsilon}\} = \sum_h \{f^{hq}\} \dot{\alpha}^{hq} = [f^q]\{\dot{\alpha}^q\} \begin{cases} \dot{\alpha}^{hq} \geq 0 & \text{if} \quad 0 = c^{hq} - \{f^{hq}\}^T \{\sigma\} \\ \dot{\alpha}^{hq} = 0 & \text{if} \quad 0 < c^{hq} - \{f^{hq}\}^T \{\sigma\}. \end{cases} \tag{12}$$

Equation (12) states that the strain velocity vector $\{\dot{\varepsilon}\}$ has to be a linear combination (with non-negative but otherwise arbitrary multipliers $\dot{\alpha}^{hq}$) of the vectors $\{f^{hq}\}$ normal to the faces of the qth yield polyhedron, which are reached by the stress vector $\{\sigma\}$.

The rate of dissipation D^q within ΔV^q (total volume $V = \sum_q \Delta V^q$) is given by:

$$D^q = \iiint_{\Delta V^q} \dot{\varepsilon}_{ij} \sigma_{ij} \, \mathrm{d}V = \iiint_{\Delta V^q} \{\dot{\varepsilon}\}^T \{\sigma\} \, \mathrm{d}V \tag{13}$$

or, using equation (12):

$$D^q = \sum_h \iiint_{\Delta V^q} \dot{\alpha}^{hq} \{f^{hq}\}^T \{\sigma\} \,\mathrm{d}V. \tag{14}$$

Considering that $\dot{\alpha}^{hq}$ can only be different from zero where $\{f^{hq}\}^T\{\sigma\}$ equals c^{hq} the integral (14) is transformed to:

$$D^q = \sum_h c^{hq} \iiint_{\Delta V^q} \dot{\alpha}^{hq} \,\mathrm{d}V = \sum_h c^{hq} \dot{\beta}^{hq} \tag{15}$$

the non-negative $\dot{\beta}^{hq}$-variables being defined by:

$$\dot{\beta}^{hq} = \iiint_{\Delta V^q} \dot{\alpha}^{hq} \,\mathrm{d}V \geq 0. \tag{16}$$

The introduction of the $\dot{\beta}^{hq}$-variables allows the evaluation of the total rate of dissipation D by the following simple linear function:

$$D = \sum_q D^q = \sum_q \sum_h c^{hq} \dot{\beta}^{hq} = \sum_q \{c^q\}^T \{\dot{\beta}^q\}. \tag{17}$$

Numerical values for the $\dot{\beta}^{hq}$-variables are obtained by solving the linear program. It is, therefore, important to understand their physical meaning: $\dot{\beta}^{hq}$ is a generalized strain velocity parameter corresponding to an average value, within ΔV^q, of the component of the strain velocity vector $\{\dot{\varepsilon}\}$ in the direction normal to the hth face of the qth yield polyhedron, i.e. in the direction of $\{f^{hq}\}$. It should also be noted that the regions of the rigid-plastic body with volume $\Delta V^q (q = 1\text{–}Q)$ are not "finite elements". Their boundaries are generally undefined. No integration over ΔV^q has ever to be performed.

4. PARAMETRIC STRESS AND DISPLACEMENT FIELDS

The suggested numerical methods require the assumption of parametric functions for both the displacement components u_i and the stress components σ_{ij}:

$$u_i = \Phi_{im}(x_1, x_2, x_3) W_m \tag{18}$$

$$\sigma_{ij} = \Psi_{ijn}(x_1, x_2, x_3) S_n \tag{19}$$

Φ_{im} and Ψ_{ijn} are the assumed functions, W_m and S_n the corresponding displacement and stress parameters ($m = 1\text{–}M$; $n = 1\text{–}N$).

Both the Φ_{im}– and the Ψ_{ijn}–functions are constructed by subdividing the continuum into E "finite elements" with simple geometric shapes. Within an element e ($e = 1-E$), the displacement and the stress components are given by:

$$u_i = \varphi^e_{ik}(x_1, x_2, x_3) w^e_k \tag{20}$$

$$\sigma_{ij} = \psi^e_{ijl}(x_1, x_2, x_3) s^e_l \tag{21}$$

where the Φ^e_{ik}'s and the ψ^e_{ijl}'s are locally assumed simple continuous functions ($k = 1\text{–}K^e$; $l = 1\text{–}L^e$). The w^e_k's and the s^e_l's are the corresponding local displacement and stress parameters. These parameters either correspond to the values of the assumed functions at specific points on the element boundary (external parameters) or are associated with the element itself (internal parameters). External parameters are always introduced in order to satisfy continuity requirements between elements as well as boundary conditions.

Local parameters w_k^e and s_l^e and global parameters W_m and S_n are related by the following topological equations:

$$w_k^e = a_{km}^e W_m \tag{22}$$

$$s_l^e = b_{ln}^e S_n \tag{23}$$

or, in matrix notation:

$$\{w^e\} = [a^e]\{W\} \tag{24}$$

$$\{s^e\} = [b^e]\{S\} \tag{25}$$

the coefficients a_{km}^e and b_{ln}^e of the boolean matrices $[a^e]$ and $[b^e]$ being defined by:

$$a_{km}^e = 1 \quad \text{if} \quad w_k^e = W_m \quad \text{otherwise} \quad a_{km}^e = 0 \tag{26}$$

$$b_{ln}^e = 1 \quad \text{if} \quad s_l^e = S_n \quad \text{otherwise} \quad b_{ln}^e = 0. \tag{27}$$

Within an element e the global and the local assumed functions are then related by:

$$\Phi_{im} = \varphi_{ik}^e a_{km} \tag{28}$$

$$\Psi_{ijn} = \psi_{ijl}^e b_{ln}. \tag{29}$$

5. LIMIT ANALYSIS BY THE LOWER BOUND APPROACH: $\lambda \rightarrow$ MAXIMUM

According to the lower bound theorem of the plasticity theory, the assumed stress components σ_{ij} must be in equilibrium with the external loads λg_i and λt_i and satisfy the yield condition (1) everywhere within the continuum.

The principle of virtual displacements states that the stresses σ_{ij} and the loads λg_i and λt_i are in equilibrium, if for any arbitrary kinematically admissible or non admissible virtual displacement field u_i^* the following variational equation is satisfied:

$$\iiint_V \frac{1}{2}(u_{i,j}^* + u_{j,i}^*)\sigma_{ij}\,\mathrm{d}V - \lambda\left[\iiint_V u_i^* g_i\,\mathrm{d}V + \iint_{S_t} u_i^* t_i\,\mathrm{d}S\right] = 0. \tag{30}$$

The first integral represents the internal virtual work done by the stresses σ_{ij}, the second and third integrals the virtual work done by the external loads λg_i and λt_i.

Introducing the parametric assumptions (18) and (19) for both the stress components σ_{ij} and the virtual displacement components u_i^* equation (30) is transformed to:

$$W_m^*\left(\left[\iiint_V \frac{1}{2}(\Phi_{im,j} + \Phi_{jm,i})\Psi_{ijn}\,\mathrm{d}V\right]S_n - \lambda\left[\iiint_V \Phi_{im}g_i\,\mathrm{d}V + \iint_{S_t} \Phi_{im}t_i\,\mathrm{d}S\right]\right) = 0. \tag{31}$$

Because equation (31) has to be satisfied by any value of the virtual displacement parameters W_1^*–W_M^* the following system of linear equations for the unknown stress parameters S_1–S_N is obtained:

$$G_{mn}S_n - \lambda P_m = 0 \qquad (m = 1\text{–}M) \tag{32}$$

or, in matrix notation:

$$[G]_{M\times N}\{S\}_{N\times 1} - \lambda\{P\}_{M\times 1} = 0 \tag{33}$$

the coefficients G_{mn} and P_m being defined by:

$$G_{mn} \equiv \iiint\limits_V \frac{1}{2}(\Phi_{im,j} + \Phi_{jm,i})\Psi_{ijn}\,\mathrm{d}V \tag{34}$$

$$P_m \equiv \iiint\limits_V \Phi_{im}g_i\,\mathrm{d}V + \iint\limits_{S_t} \Phi_{im}t_i\,\mathrm{d}S. \tag{35}$$

$[G]$ will be called the global equilibrium matrix, $\{P\}$ the global load vector of the system. A coefficient G_{mn} represents the work done by the stresses due to $S_n = 1$ for the strains due to $W_m^* = 1$. P_m represents the work done by the external loads for the displacements due to $W_m^* = 1$. Obviously equation (33) will in general only lead to an approximate satisfaction of the equilibrium conditions.

Equation (34) can be used to evaluate the G_{mn}-coefficients as long as the assumed displacement functions Φ_{im} are kinematically admissible, i.e. continuous, and satisfy the kinematic boundary conditions:

$$u_i^* = \Phi_{im} = 0 \quad \text{on} \quad S_u. \tag{36}$$

However, provided that the Ψ_{ijn}-functions satisfy certain continuity and boundary conditions (see Section 8) the integral (34) can be evaluated even if the Φ_{im}-functions are not kinematically admissible by the following transformation (Green's theorem):

$$G_{mn} = -\iiint\limits_V \Phi_{im}\Psi_{ijn,j}\,\mathrm{d}V + \iint\limits_{S_u+S_t} \Phi_{im}\Psi_{ijn}\nu_j\,\mathrm{d}S \tag{37}$$

the ν_j's being the components of a unit vector $\{\nu\}$ normal to the boundary surface (see Fig. 1).

The stress parameters S_1–S_N also have to satisfy yield conditions. These are formulated at Q checkpoints, where the functions Ψ_{ijn} assume the values $\Psi_{ijn}^q (q = 1\text{–}Q)$:

$$\sigma_{ij}^q = \Psi_{ijn}^q S_n \tag{38}$$

or, in matrix notation:

$$\{\sigma^q\} = [\Psi^q]\{S\}. \tag{39}$$

Introducing equation (39) in (6) the following system of linear inequalities results:

$$0 \le \{c^q\} - [f^q]^T[\Psi^q]\{S\} \qquad (q = 1\text{–}Q). \tag{40}$$

The yield conditions can be satisfied exactly everywhere inside the continuum only if the assumed Ψ_{ijn}-functions are piecewise linear or constant. If nonlinear Ψ_{ijn}-functions are chosen, local violations of the yield conditions have to be taken into account.

From (33) and (40), together with the condition λ = maximum, the following linear program is obtained (see Fig. 3):

$$\left.\begin{aligned} &\lambda \to \text{maximum} \\ &0 = -\{P\}\lambda + [G]\{S\} \\ &0 \leq \{c^q\} - [f^q]^T[\Psi^q]\{S\} \qquad (q = 1\text{–}Q) \end{aligned}\right\}. \tag{41}$$

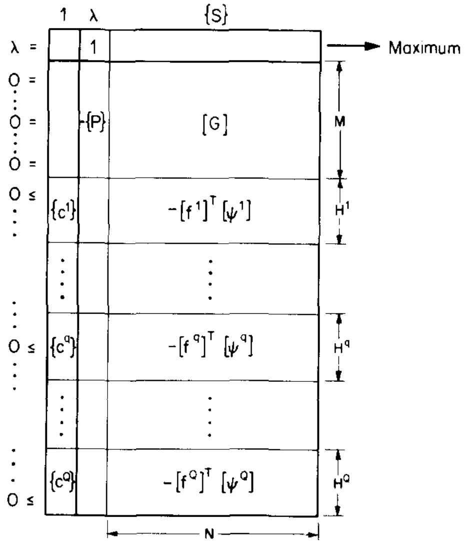

FIG. 3. Tableau form of the linear program (41).

6. LIMIT ANALYSIS BY THE UPPER BOUND APPROACH: λ → MINIMUM

The rate of work L, done by the external loads during collapse, can be evaluated as follows using, for the displacement velocity field $\dot{u}_i$, the assumptions (18) and the definition (35):

$$L = \iiint_V \dot{u}_i \lambda g_i \, \mathrm{d}V + \iint_{S_t} \dot{u}_i \lambda t_i \, \mathrm{d}S$$

$$= \lambda \left[\iiint_V \Phi_{im} g_i \, \mathrm{d}V + \iint_{S_t} \Phi_{im} t_i \, \mathrm{d}S \right] \dot{W}_m = \lambda P_m \dot{W}_m = \lambda \{P\}^T \{\dot{W}\}. \tag{42}$$

The internal rate of dissipation D is given by equation (17):

$$D = \sum_q D^q = \sum_q \sum_h c^{hq} \dot{\beta}^{hq} = \sum_q \{c^q\}^T \{\dot{\beta}^q\}. \tag{43}$$

Because only the ratio between L and D is relevant, the following condition can be imposed:

$$\{P\}^T \{\dot{W}\} = 1. \tag{44}$$

During collapse L equals D. It follows:

$$L = \lambda\{P\}^T\{\dot{W}\} = \lambda = D = \sum_q \{c^q\}^T\{\dot{\beta}^q\} \rightarrow \text{minimum}. \tag{45}$$

The upper-bound theorem of the plasticity theory requires the velocity field $\dot{u}_i$ used to evaluate L to be kinematically compatible with the strain velocity field $\dot{\varepsilon}_{ij}$ used to evaluate D. Kinematic compatibility equations between the displacement velocity parameters $\dot{W}_m$ and the generalized strain velocity parameters $\dot{\beta}^{hq}$ have therefore to be formulated.

The principle of virtual stresses states, that a displacement velocity field $\dot{u}_i$ and a strain velocity field $\dot{\varepsilon}_{ij}$ are kinematically compatible if for any arbitrary virtual stress field σ^*_{ij} the following variational equation is satisfied [11]:

$$\iiint\limits_V \sigma^*_{ij}\dot{\varepsilon}_{ij}\,\mathrm{d}V - \iiint\limits_V g^*_i\dot{u}_i\,\mathrm{d}V - \iint\limits_{S_t} t^*_i\dot{u}_i\,\mathrm{d}S = 0. \tag{46}$$

The first integral represents the rate of work done by the virtual stresses σ^*_{ij}. The second and third integrals represent the rate of work done by virtual body forces g^*_i and virtual surface tractions t^*_i. The virtual loads g^*_i and t^*_i must build, together with the σ^*_{ij}'s, an equilibrium system, i.e. they are derived from the virtual stresses σ^*_{ij} by the equilibrium equations:

$$g^*_i = -\sigma^*_{ij,j} \tag{47}$$

$$t^*_i = v_j\sigma^*_{ij}. \tag{48}$$

Introducing the parametric functions (18) and (19) for the $\dot{u}_i$'s and σ^*_{ij}'s equation (46) becomes:

$$S^*_n\left(\iiint\limits_V \Psi_{ijn}\dot{\varepsilon}_{ij}\,\mathrm{d}V - \left[-\iiint\limits_V \Psi_{ijn,j}\Phi_{im}\,\mathrm{d}V + \iint\limits_{S_t} \Psi_{ijn}v_j\Phi_{im}\,\mathrm{d}S\right]\dot{W}_m\right) = 0. \tag{49}$$

This equation has to be satisfied for any value of the virtual stress parameters S^*_1–S^*_N. Remembering the definition (37) the following system of kinematic compatibility equations is obtained:

$$\iiint\limits_V \Psi_{ijn}\dot{\varepsilon}_{ij}\,\mathrm{d}V - G_{mn}\dot{W}_m = 0 \qquad (n = 1\text{–}N). \tag{50}$$

The volume integral of equation (50) can be transformed by subdividing the domain of integration V in Q parts ΔV^1–ΔV^Q. If everywhere within $\Delta V^q (q = 1\text{–}Q)$ the yield conditions are given by the qth yield polyhedron, the following transformation holds:

$$\iiint\limits_V \Psi_{ijn}\dot{\varepsilon}_{ij}\,\mathrm{d}V = \sum_q \iiint\limits_{\Delta V^q} \Psi_{ijn}\dot{\varepsilon}_{ij}\,\mathrm{d}V = \sum_q\sum_h f^{hq}_{ij} \iiint\limits_{\Delta V^q} \Psi_{ijn}\dot{\alpha}^{hq}\,\mathrm{d}V. \tag{51}$$

To further transform this integral an approximation is necessary:

$$\iiint\limits_{\Delta V^q} \Psi_{ijn}\dot{\alpha}^{hq}\,\mathrm{d}V \approx \Psi^q_{ijn}\iiint\limits_{\Delta V^q} \dot{\alpha}^{hq}\,\mathrm{d}V = \Psi^q_{ijn}\dot{\beta}^{hq} \tag{52}$$

where Ψ^q_{ijn} represents the value of the function Ψ_{ijn} at a checkpoint q. The generalized strain velocity parameters $\dot{\beta}^{hq}$ are defined by (16).

Introducing equations (51) and (52) in (50), the following system of linear kinematic compatibility equations for the $\dot{W}_m$- and the $\dot{\beta}^{hq}$-parameters is obtained:

$$-G_{mn}\dot{W}_m + \sum_q \sum_h \Psi^q_{ijn} f^{hq}_{ij} \dot{\beta}^{hq} = 0 \qquad (n = 1\text{–}N) \tag{53}$$

or, in matrix notation:

$$-[G]^T\{\dot{W}\} + \sum_q [\Psi^q]^T [f^q]\{\dot{\beta}^q\} = 0. \tag{54}$$

Obviously, because of the transformation (52), kinematic compatibility conditions are only enforced in an approximate way, unless piecewise constant Ψ_{ijn}-functions within each of the ΔV^q's are assumed.

From (45), (44) and (54) the following linear program for the unknown parameters $\dot{W}_m$ and $\dot{\beta}^{hq}$ is obtained (see Fig. 4)

$$\left.\begin{aligned} \lambda &= \sum_q \{c^q\}^T\{\dot{\beta}^q\} \rightarrow \text{minimum} \\ 0 &= 1 - \{P\}^T\{\dot{W}\} \\ 0 &= -[G]^T\{\dot{W}\} + \sum_q [\Psi^q]^T[f^q]\{\dot{\beta}^q\} \\ &\{\dot{\beta}^q\} \geq 0 \qquad (q = 1\text{–}Q) \end{aligned}\right\}. \tag{55}$$

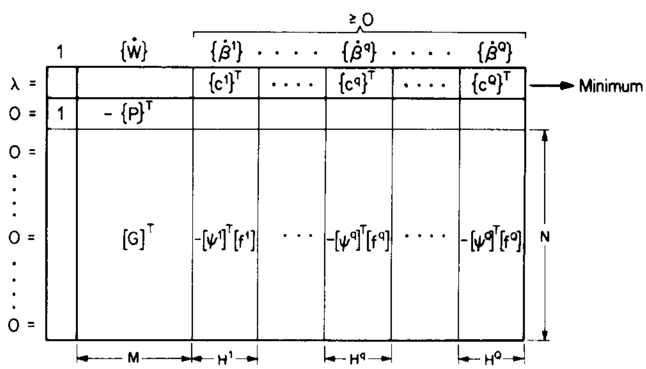

FIG. 4. Tableau form of the linear program (55).

The linear programs (41) and (55) are dual to each other. The same load factor λ will, therefore, be obtained. As expected the value of λ only depends on the choice of the mathematical model, not on the method of solution used (lower-bound or upper-bound approach), provided that the approximations necessary for both methods are introduced in a consistent way. A lower bound of the true value of λ is obtained, if the assumed functions Ψ_{ijn} and the linear inequalities (40) guarantee that the yield condition (1) is everywhere satisfied within the continuum and if the external loads are such that microscopic equilibrium conditions are nowhere violated. An upper bound (at least for the linearized yield conditions) is obtained if the transformation (52) is valid without approximation.

By solving one of the linear programs (41) or (55) the solution of the other is also known. Numerical values not only for λ but also for the S_n-, $\dot{W}_m$- and $\dot{\beta}^{hq}$-parameters are, therefore, obtained.

The displacement velocity parameters $\dot{W}_1$–$\dot{W}_M$ describe the collapse mechanism. The stress parameters S_1–S_N define a corresponding state of admissible stresses. However, because this is defined in an unique way only in the regions and in the directions, in which plastic flow occurs; the values of the S_n-parameters will generally not be very meaningful, as large portions of the continuum may remain rigid during collapse. The generalized strain velocity parameters $\dot{\beta}^{hq}$ can be used to check the regions and the directions of plastic flow.

7. ON THE ASSEMBLAGE OF [G] and {P}

By means of equations (28) and (29) the coefficients G_{mn} and P_m can be evaluated as a sum over E finite elements with volume V^e and surface $S^e(e = 1\text{–}E)$:

$$G_{mn} = \sum_e \iiint_{V^e} \frac{1}{2}(\Phi_{im,j}+\Phi_{jm,i})\Psi_{ijm}\,\mathrm{d}V$$

$$= \sum_e a^e_{km}\left[\iiint_{V^e} \frac{1}{2}(\varphi^e_{ik,j}+\varphi^e_{jk,i})\psi_{ijl}\right]b^e_{ln} = \sum a^e_{km}g^e_{kl}b^e_{ln} \tag{56}$$

$$P_m = \sum_e a^e_{km}\left[\iiint_{V^e} \varphi^e_{ik}g_i\,\mathrm{d}V + \iint_{S^e} \varphi^e_{ik}t_i\,\mathrm{d}S\right] = \sum_e a^e_{km}p^e_k\,. \tag{57}$$

In matrix notation:

$$[G] = \sum_e [a^e]^T[g^e][b^e] \tag{58}$$

$$\{P\} = \sum_e [a^e]^T\{p^e\} \tag{59}$$

where the coefficients g^e_{kl} and p^e_k of the "local" equilibrium matrices $[g^e]$ and the coefficients of the "local" load vectors $\{p^e\}$ can be defined as follows, provided that the assumed displacement functions are kinematically admissible:

$$g^e_{kl} \equiv \iiint_{V^e} \varphi^e_{ik}g_i\,\mathrm{d}V + \iint_{S^e} \varphi^e_{ik}t_i\,\mathrm{d}S \tag{60}$$

$$p^e_k \equiv \iiint_{V^e} \varphi^e_{ik}g_i\,\mathrm{d}V + \iint_{S^e} \varphi^e_{ik}t_i\,\mathrm{d}S. \tag{61}$$

If the assumed displacement functions are not conforming, work is done on the element boundaries so that different definitions have to be used. However, bearing in mind the physical meaning of the global G_{mn}- and P_m-coefficients, analytical expressions for the g^e_{kl}- and p^e_k- are generally easy to derive once the local φ^e_{ik}- and ψ^e_{ijl}-functions are chosen.

The global equilibrium matrix $[G]$ and the global load vector $\{P\}$ are then assembled by the summations procedures described by equations (58) and (59). The similarity between these procedures and the well known "direct stiffness method" of elastic analysis is evident.

8. ON THE CHOICE OF THE FINITE ELEMENT MODEL

In choosing the finite element approximation for both the stress and the displacement fields, the first important question arising concerns the necessary continuity and boundary conditions, which have to be satisfied a priori by the assumed Φ_{im}- and Ψ_{ijn}-functions.

Virtual work principles can be applied as long as the integrals (34) or (37) can be evaluated. This is possible, and the transformation from (34) to (37) is valid, if the following conditions are satisfied.

Let us consider a point on the interface between two elements, where discontinuities may occur. A cartesian coordinate system $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is defined; the $\bar{x}_1$- and $\bar{x}_2$-axes lying in a plane tangent to that surface, the $\bar{x}_3$-axis being normal to it. $\bar{u}_i$ and $\bar{\sigma}_{ij}$ represent the corresponding stress and displacement components.

If the assumed finite element displacement field is such that discontinuities of the displacement component $\bar{u}_i$ $(i = 1, 2, 3)$ are possible, the assumed stress field must guarantee the continuity of the stress component $\bar{\sigma}_{3i}$. If $\bar{u}_i$ is continuous, no continuity is required for $\bar{\sigma}_{3i}$. For $\bar{\sigma}_{1i}$ and $\bar{\sigma}_{2i}$ continuity is never required.

Boundary conditions can be regarded as a special kind of continuity conditions: if the continuity of a displacement component $\bar{u}_i$ is ensured between elements, the geometric boundary condition:

$$\bar{u}_i = 0 \quad \text{on } S_u \tag{62}$$

has to be enforced. If $\bar{u}_i$ is discontinuous, the equilibrium boundary condition:

$$\bar{\sigma}_{3i} = \lambda \bar{t}_i \quad \text{on } S_t \tag{63}$$

has to be enforced, $\bar{t}_i$ being the specified surface traction in the direction of $\bar{x}_i$. In evaluating the P_m-coefficients by equation (35), only body forces g_i have then to be taken into account.

In constructing parametric fields by the finite element method, continuity requirements are always met by introducing as function parameters element-boundary-values of the function itself. Therefore, provided that the proper continuity conditions between elements are satisfied, the enforcement of boundary conditions is never difficult, the boundary values of the stress and displacement components, for which boundary conditions have to be enforced, being always specified by boundary parameters alone.

A second important question arising concerns the ratio N/M between the number N of stress parameters and the number M of displacement parameters.

For a given mesh N and M depend from the chosen finite element stress and displacement models and also from the boundary conditions of the problem. It can be shown, however, that for very fine meshes, i.e. if the number of elements goes to infinity, the ratio N/M only depends from the chosen finite element models and is, therefore, independent from the considered problem.

Let us for instance consider such an infinitely fine two-dimensional triangular element mesh. If NE is the number of elements, NJ the number of joints and NS the number of sides of the mesh, the following relations hold:

$$NE/NJ = 2 \tag{64}$$

$$NS/NJ = 3. \tag{65}$$

For quadrilateral element meshes:

$$NE/NJ = 1 \tag{66}$$

$$NS/NJ = 2. \tag{67}$$

As the finite elements parameters are always associated either with the joints or with the elements or with the sides of the mesh, the ratio N/M can readily be determined.

The N stress parameters S_1–S_N must satisfy M linear equilibrium equations, one for each of the assumed displacement parameters W_1–W_M. Obviously N has to be greater than M, if the system of linear equations (32) has to have a solution. If N equals M, the system is statically determined, which is certainly unacceptable for continuous structures. On the other hand N should not be too large compared to M, as equilibrium conditions may then be badly violated.

But how large should N/M be in order to satisfy equilibrium within the continuum in a consistent way? A clear cut answer is not easy to find, however, and the following criterium seems reasonable and has been followed in the assumption of different successful finite element models: the ratio N/M for an infinitely fine mesh should equal the ratio N_c/M_c of the corresponding continuous problem, where N_c is the number of independent stress components and M_c the number of independent displacement components within the continuum. M_c is also the number of partial differential equations, which have to be satisfied by the N_c independent stress components (for general three-dimensional problems: $N_c/M_c = 6/3$; for axisymmetric problems: $N_c/M_c = 4/2$; for plate-stretching problems: $N_c/M_c = 3/2$; for plate-bending problems: $N_c/M_c = 3/1$).

9. TWO PLATE-BENDING MODELS

Two plate-bending triangular-element models have been implemented. Both models were originally proposed by Herrmann [4, 5] in "mixed" formulations of elastic finite element plate-bending analysis. The first (linear–linear) model assumes linear deflections and linear bending moment distribution, the second (linear–constant) model assumes linear deflection and constant moment distribution within each triangular element. In both cases the reinforced concrete Wolfensberger's yield conditions (9) are used.

The displacement and stress parameters of the linear–linear model are the plate deflections w and the three moments m_x, m_y and m_{xy} at each joint of the mesh. While deflection continuity is ensured by the assumed linear functions, kinematically non-admissible slope discontinuities along the sides of the mesh occur. The continuity of the corresponding stresses, i.e. of the normal moments m_n along each side is, therefore, necessary. The assumed functions guarantee the continuity of all three moments m_x, m_y, m_{xy}, thus also of the normal moments m_n. However, this kind of continuity is excessive and actually represents a drawback of the model, as tangential and twisting moments do not need to be continuous across element interfaces.

Kinematic boundary conditions for the plate deflections w (not for the slopes, which are discontinuous between elements) and static boundary conditions for the normal moments m_n (not for the shear forces, which are discontinuous and not for the twisting moments, whose continuity is unnecessary) have to be enforced by eliminating from the linear program the corresponding boundary parameters.

Each triangular element is associated with three displacement parameters (w at each vertex) and nine stress parameters (m_x, m_y, m_{xy} at each vertex). Considering that, when displacing the assembled element mesh, work is done only along the element edges, the local equilibrium matrix $[g^e]_{3\times 9}$ is easy to derive (see Fig. 5):

$$[g^e]_{3\times 9} = \frac{1}{2}\begin{bmatrix} -\frac{l_1}{h_1} & \frac{a_1}{h_1} & \frac{b_1}{h_1} \\ \frac{b_2}{h_2} & -\frac{l_2}{h_2} & \frac{a_2}{h_2} \\ \frac{a_3}{h_3} & \frac{b_3}{h_3} & -\frac{l_3}{h_3} \end{bmatrix}_{3\times 3} \cdot \begin{bmatrix} 0 & \{r_1\}^T & \{r_1\}^T \\ \{r_2\}^T & 0 & \{r_2\}^T \\ \{r_3\}^T & \{r_3\}^T & 0 \end{bmatrix}_{3\times 9}. \tag{68}$$

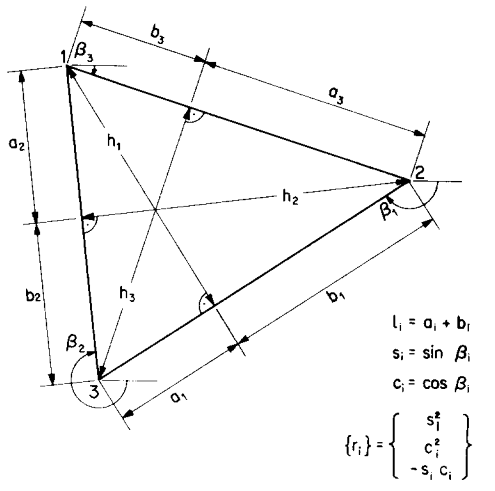

FIG. 5. Triangular element.

If boundary conditions are ignored, a mesh with NJ joints has NJ displacement parameters ($M = NJ$) and $3{\cdot}NJ$ stress parameters. The ratio $N/M = 3$ corresponds to the ratio N_c/M_c for nondiscrete plate-bending problems as explained in the previous section.

Yield conditions are checked at each joint (number of checkpoints $Q = NJ$) resulting in $8 \cdot Q = 8 \cdot NJ$ linear inequalities. While this generally ensures that the yield conditions are everywhere satisfied (moments vary linearly between checkpoints), no bound of the true value of λ is found as both equilibrium and kinematic compatibility conditions may be locally violated.

The linear–constant model uses the same displacement assumptions but the moment distribution is constant within each element. The normal moments m_n along each mesh side are chosen as stress parameters. The state of stress inside each element is, therefore, defined by the three normal moments m_{n1}, m_{n2}, m_{n3}, which are linearly related to the moments m_x, m_y, m_{xy} as follows (see Fig. 5):

$$\begin{Bmatrix} m_x \\ m_y \\ m_{xy} \end{Bmatrix} = \begin{bmatrix} s_1^2 & c_1^2 & -2s_1c_1 \\ s_2^2 & c_2^2 & -2s_2c_2 \\ s_3^2 & c_3^2 & -2s_3c_3 \end{bmatrix}^{-1} \begin{Bmatrix} m_{n1} \\ m_{n2} \\ m_{n3} \end{Bmatrix} = \begin{bmatrix} \{r_1\}^T \\ \{r_2\}^T \\ \{r_3\}^T \end{bmatrix}^{-1} \begin{Bmatrix} m_{n1} \\ m_{n2} \\ m_{n3} \end{Bmatrix}. \tag{69}$$

This choice of the stress parameters ensures the necessary continuity of normal stresses across element interfaces, while no excessive continuity is present. Again deflection- and normal-moment-boundary conditions have to be imposed.

Local equilibrium matrices $[g^e]_{3\times 3}$ are defined by:

$$[g^e]_{3\times 3} = \begin{bmatrix} -\dfrac{l_1}{h_1} & \dfrac{a_1}{h_1} & \dfrac{b_1}{h_1} \\ \dfrac{b_2}{h_2} & -\dfrac{l_2}{h_2} & \dfrac{a_2}{h_2} \\ \dfrac{a_3}{h_3} & \dfrac{b_3}{h_3} & -\dfrac{l_3}{h_3} \end{bmatrix}_{3\times 3} . \tag{70}$$

Fine triangular meshes have about three times as many sides as joints ($NS \simeq 3 \cdot NJ$). The ratio N/M between the number of stress parameters $N \simeq NS \simeq 3 \cdot NJ$ and the number of displacement parameters $M \simeq NJ$ corresponds therefore to the N_c/M_c ratio.

Yield conditions are checked in each element ($Q = NE$), resulting in about twice as many linear inequalities as in the linear–linear model ($NE \simeq 2 \cdot NJ$).

The main advantage of the linear-constant model is, that a lower bound of the true value of λ is found, provided that the plate is loaded only by concentrated loads acting at the joints. According to the classical Kirchhoff's theory, twisting moments do not need to be continuous nor need they to satisfy static boundary conditions, as long as the external loads are in equilibrium with the "Kirchhoff's shear forces". It is then easy to see that concentrated joint loads are balanced by internal concentrated forces at both ends of each side, arising from twisting moment discontinuities along the side. Equilibrium and yield conditions being satisfied exactly, a lower bound for λ is found.

10. SOLUTION ALGORITHMS AND NUMERICAL RESULTS

For the plate-bending models described in the previous section two separate FORTRAN IV programs were written and tested on the CDC-6500/CDC-6400 double system of the Computer Center of the Swiss Federal Institute of Technology in Zürich.

While mass storage is used to partition the program itself and for several auxiliary data transfers, the main optimalisation is done in core. However, in order to reduce both storage requirements and central processor time a sophisticated modified version of the so-called revised simplex algorithm is used. The main idea of this algorithm is to recalculate at each simplex optimalisation step all needed coefficients from the initial data of the problem and from an auxiliary "basis matrix", thus never having to store the full matrix of the linear program.

A good measure of storage requirement is given by the number $N \cdot (N - M)$ of computer words needed to store the basis matrix. This number is independent from the number of checkpoints Q, i.e. from the number of inequalities used to linearize the yield conditions, and is approximately equal $6 \cdot NJ^2$ for both plate-bending models. In practice, with a maximum core size of $140{,}000_8 \cong 49{,}000_{10}$ words, meshes with *ca.* 85 joints can be handled. More details are given in [7].

Central processor time is difficult to predict being a function of the problem-dependent number of necessary optimalisation steps. An approximate empirical formula relating CP-time and number of joints NJ is given by:

$$CP \text{ (time in sec)} = (0{\cdot}3-0{\cdot}7)10^{-3} \cdot NJ^3. \tag{71}$$

Numerical results for a circular reinforced concrete plate in bending with clamped or simply supported edges under a uniformly distributed load are shown in Fig. 6. Figure 7

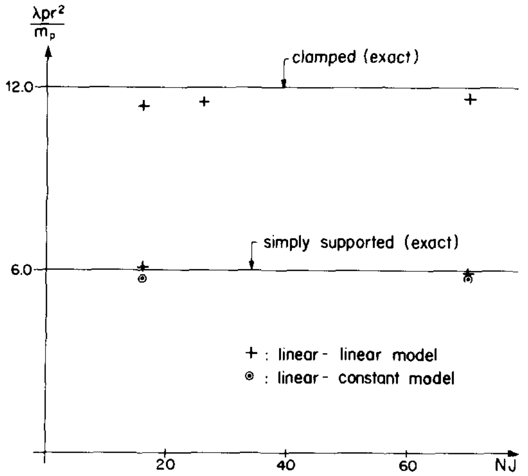

FIG. 6. Circular plate in bending (radius $= r$) with constant reinforcement $m_p = P_x = P_y = N_x = N_y$ and uniformly distributed load λp (NJ = number of joints of the triangular element mesh).

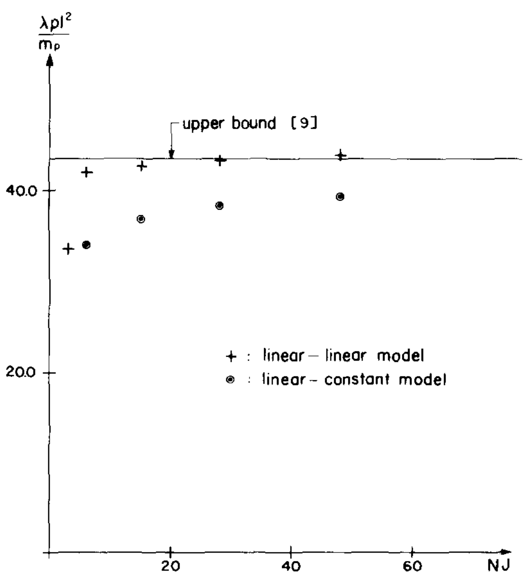

FIG. 7. Clamped square plate in bending (side length $= l$) with constant reinforcement and uniformly distributed load λp.

shows corresponding results for a square plate with all edges clamped. While for the circular plates exact solutions of the ultimate load problem are known, for the clamped square plate only an upper bound of the true solution could be determined (see [9]).

Figure 8 shows the finite element subdivision as well as the displacement velocity distribution during collapse along the lines of symmetry for an infinitely large reinforced

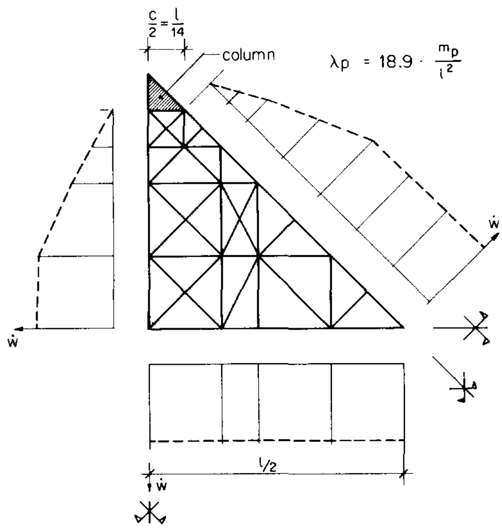

FIG. 8. Displacement velocity field pattern for continuous flat slab with constant reinforcement and uniformly distributed load λp.

concrete slab, supported by a regular mesh of square columns. The linear–linear model was used in this example. The result obtained for a column-side to span ratio $c/l = 1/7$ cannot be compared with any known result. A similar calculation for a ratio $c/l = 0$ (i.e. for point-columns) gives an ultimate load $\lambda p = 11.6 \cdot m_p/l^2$, which is in good agreement with the upper bound $\lambda p = 4 \cdot \pi \cdot m_p/l^2$ given by [13].

11. SOME PLATE-STRETCHING MODELS

Table 1 describes six possible plate-stretching triangular and quadrilateral element models. The displacement components in the direction of the coordinates x and y of a cartesian system lying in the plane of the plate are denoted by u_x and u_y, corresponding stress components by σ_x, σ_y and $\tau = \tau_{xy} = \tau_{yx}$. Normal and shear stresses along element edges are denoted by σ_n and τ_{nt}.

Within the triangular elements the assumed stress and displacement functions are constant or linear, within the quadrilateral elements constant or bilinear, i.e. linear along the element edges (see for instance [14, Section 7.2]).

The third, fourth and fifth columns of Table 1 give the shape of the assumed displacement functions within each element, the type of displacement parameters and their approximate number M for a mesh with NJ joints. The next three columns describe the assumed stress field in a similar way. The last two columns show the position of the points, where yield conditions must be checked and their approximate number Q.

TABLE 1. SIX POSSIBLE PLATE STRETCHING SOLUTIONS

No.	Element-shape	Displacement field			Stress field			Yield condition checkpoints	
		Distribution	Type of parameters	$\frac{M}{NJ}$	Distribution	Type of parameters	$\frac{N}{NJ}$	Position of checkpoints	$\frac{Q}{NJ}$
1		Linear	$u_x u_y$ at each joint	2	Linear	$\sigma_x \sigma_y \tau$ at each joint	3	At each joint	1
2		Linear	$u_x u_y$ at each joint	2	Constant	$\sigma_x \sigma_y \tau$ in each element	6	In each element	2
3		Linear	$u_x u_y$ at each joint	2	Constant	σ_n *or* τ_{nt} on each side	3	In each element	2
4		Bilinear	$u_x u_y$ at each joint	2	Bilinear	$\sigma_x \sigma_y \tau$ at each joint	3	At each joint	1
5		Bilinear	$u_x u_y$ at each joint	2	Constant	$\sigma_x \sigma_y \tau$ in each element	3	In each element	1
6		Constant	$u_x u_y$ in each element	2	Bilinear	$\sigma_x \sigma_y \tau$ at each joint	3	At each joint	1

Particularly interesting is the model No. 2, as it can be shown that the obtained value of λ must be, at least for the assumed linear yield conditions, an upper bound of the true value of the load factor. The transformation (52) is valid without approximations. This is also the only model for which the ratio $N/M = 6/2$ is larger than the ratio $N_c/M_c = 3/2$ of continuous plate-stretching problems, which is not surprising for an equilibrium-violating upper-bound type of model.

The model No. 6 (with rectangular elements only) has been successfully implemented by Vollenweider [10] for plain-strain soil mechanics limit load problems, using as yield condition a linearized form of Coulomb–Mohr's rupture hypothesis.

12. CONCLUSIONS

In finite element plastic analysis, just as in elastic analysis, "computer shapes theory". Because an efficient use of the available hardware can broaden the range of possible applications enormously, the programming techniques, rather than the purely theoretical aspects of the problem are of primary importance.

If plastic analysis has to become a widely used tool in civil engineering, like finite element elastic analysis today already is, much work is left to be done in comparing different models and solution algorithms, and in evaluating the accuracy, which can be obtained with a bearable computational effort.

An interesting possibility is to use nonlinear optimalisation procedures. Hodge and Belytschko [6] describe such a procedure for plate bending and discuss the advantages and disadvantages of nonlinear vs. linear procedures. It seems, however, that for a final conclusion more experience is needed.

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(*Received* 28 *January* 1972)

Абстракт—Работа касается разработки численных методов определения предельной нагрузки двух- и трехмерных конструкций, предполагая идеально жестко-пластическое поведение материала. В согласии с классической теорией пластичности, задача предельной нагрузки определяется математической формулой в смысле задачи вычисления максимума или минимума линейной функции, независимые переменные которой подверженные неравномерным силам связи. С целью использования метода линейного программиров ания, приближаются эти вообще нелинейные силы связи рядами ограничений линейных неравенств. Затем, можно определить линейные уравнения равновесия или кинематической совместимости, причем соответствующие коэффициенты получаются с помощью методов виртуальной работы. Это требует приема полей параметрических напряжений и перемещений, полученных с помощью метода конечного элемента. Даются численные результаты для двух моделей иегиба пластинок. Описываются и обсуждаются также возможность исследования моделей пластинок, подверженных растяжнию.